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Special Technical Report 1

FUNDAMENTAL NOTES
ON INSTABILITY ANALYSIS

By: T.N.C. WANG

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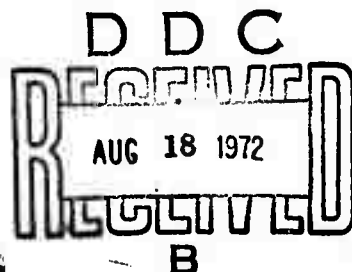
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13. ABSTRACT

In this special technical report we present fundamental studies on instability analysis. When an electromagnetic wave linearly interacts with an active medium or system, there can occur basically two types of instabilities. One of them, in which the wave grows indefinitely with time, is called absolute instability, and the other, in which the wave grows spatially as it travels, is called convective instability. By conformal mapping of the dispersion equation from ω onto the k plane or vice versa, we have made a detailed analysis of these two types of instabilities. General procedures for analyzing instabilities are summarized. Effects of driving sources and the possible occurrence of branch cuts, in the k plane, of the dispersion equation are also discussed.

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I INTRODUCTION

In recent years the geophysical scientific community has shown considerable interest in the idea of magnetospheric plasma-injection experiments. It is hoped that these experiments will aid in the study of the following important problems:

(1) Particle precipitation and the physics of the lower ionosphere.

(2) The details of wave-particle interactions, which lead to the full understanding of the mechanisms causing natural emissions and the control of artificially triggered emissions.

(3) The physics of aurorae.

(4) The feasibility of using an appropriate controllable amplified wave for space-to-space and/or space-to-ground communications.

In all the above problem areas, it is of primary importance to understand the details of wave-particle/medium interactions and the related instabilities.

When a wave interacts with a linearized system, two fundamental types of instability (absolute and convective) can occur. The detailed distinctions between the two are given in Section III. Due to various applications of instability theories in both the space experiments and the laboratory plasma devices, such as fusion and beam-plasma devices, the subject of wave-plasma instability has been investigated by a great number of workers (listed in the Bibliography). The instability analysis and criteria and the computation for the various growth rates have been fairly well developed. However, in most of the earlier articles on the subject, the authors give only brief analyses on instability theory. Also, the

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theorems and the analyses for determining the nature of instabilities are scattered in the various journal articles and reports. It is the purpose of this report to present an integrated, detailed study on the subject of instability analysis. Starting from the very elementary analysis, we proceed step by step toward the conclusions on the theory of instabilities.

It is hoped that the present report can be of use to part of the geophysical community in understanding the subject of wave-plasma (particle) instabilities and their related applications in the space plasmas.

We do not cite the references in the various sections. The important publications used in the preparation of the present report are listed in the Bibliography.

II   DISPERSION RELATION

When we consider the problem of electromagnetic radiation or wave propagation in a uniform, time-invariant medium, we can apply the Fourier transform to both time and spatial variables. Using linearized source-free Maxwell's equations combined with the appropriate dynamic relations to determine the constituent parameters such as permittivity, ϵ, permeability, μ, and conductivity, σ, we can derive the so-called dispersion equation $D(\omega, \vec{k}) = 0$. What we customarily know of the dispersion equation is the following:

(1)   The solutions of $D(\omega, \vec{k}) = 0$ determine the characteristic wave modes of the system. When both ω and $\vec{k}$ in these solutions are real, the characteristic modes are then representable by steady-state plane waves, with their phase varying as $\exp \pm j(\omega t - \vec{k} \cdot \vec{x})$.

(2)   In a linear regime, when we perturb the system by a small source, the response $r(\vec{x}, t)$ can be expressed by the superposition of the characteristic wave modes.

(3)   Whenever $D(\omega, \vec{k})$ is an algebraic polynomial of finite order, the system can support only a finite number of characteristic modes. On the other hand, when $D(\omega, \vec{k})$ is a transcendental function or an integral form (e.g., Landau waves, cyclotron harmonic waves, Bernstein modes in warm plasma) the system may support an infinite number of characteristic modes. The solutions $\omega_n(\vec{k})$--here, both ω_n and $\vec{k}$ are assumed to be real--are classified in terms of the various branches in the dispersion diagram (ω vs. $\vec{k}$ plots). An essential feature to note is that the modes $\omega_n(\vec{k})$, for the same n and different $\vec{k}$'s, as well as for different n's, are

linearly independent. Alternatively, we can describe the wave types $\omega_n(\vec{k})$ by the solutions $\vec{k}_n(\omega)$. For the case of both ω and $\vec{k}$, considered here to be real, there is a one-to-one correspondence between the branches $\omega_n(\vec{k})$ and $\vec{k}_n(\omega)$.

(4) The phase velocity $\vec{V}_p = (\omega/k)\hat{k}$ and the group velocity $\vec{v}_g = \nabla_{\vec{k}}\omega$ for the characteristic waves can be directly derived from the dispersion equation.

The above understanding of the dispersion equation is sufficient for analyzing the waves in simple systems involving passive media. Since in these simple systems it is not possible for the waves to draw the energy from the medium or the system, the situation for the wave system is stable. In a stable system, the characteristic modes determined from the dispersion equation are either steady-static propagating plane waves (both ω and $\vec{k}$ are real) or attenuated plane waves (ω real, $\vec{k}$ complex).

When we deal with electromagnetic problems in a complex system involving active media such as warm plasmas and energetic beams, it is important for us to know the further implications as well as mathematical properties of the dispersion equation.

In the abovementioned complex system it is possible for the waves to draw energy from the medium or system, causing the wave perturbation to grow. The wave system is thus referred to as an unstable system. In analyzing the wave properties in an unstable system, several additional implications of the dispersion equation $D(\omega,\vec{k}) = 0$ should be noted:

(1) $D(\omega,\vec{k})$ is regarded as a function of two complex variables, ω, $\vec{k}$. It should be noted that in general there is no one-to-one correspondence between the branches $\vec{k}_n$ (ω real) and ω_n ($\vec{k}$ real). However, a connection can be made through a process of conformal mapping in which ω and $\vec{k}$ are regarded as complex variables.

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(2) For an active system, there arises a difficulty in interpreting the roots with $\vec{k}$ complex derived from solving $D(\omega,\vec{k}) = 0$ directly. This is because we cannot distinguish, purely from the sign of Im $\vec{k}$ in exp $\pm j(\omega t - \vec{k} \cdot \vec{x})$, whether the wave is amplifying or attenuating, since we do not know whether to consider increasing or decreasing values in spatial coordinates. For stable systems, the complex $\vec{k}$ roots cause no ambiguity because on physical grounds all such waves must be attenuating. In view of the foregoing arguments, we emphasize that, in unstable systems, solving for the roots with $\vec{k}$ complex and specified ω from $D(\omega,\vec{k}) = 0$ has little significance.

(3) The roots of $D(\omega,\vec{k}) = 0$ with Im $\omega \gtrless 0$, where ">" and "<" apply to exp $- j(\omega t - \vec{k} \cdot \vec{x})$ and exp $+ j(\omega t - \vec{k} \cdot \vec{x})$, respectively, indicate that the active system is unstable. Thus, instability in some sense does exist in the waves. An ambiguity does not arise in the complex ω roots here, as discussed in Item 2, because we always consider the increase in time (causality).

In this section we have described several basic implications of a dispersion equation. It should be mentioned that many more implications than we have listed above can be drawn from the dispersion after a detailed analysis. In fact, a sufficient analysis of $D(\omega,\vec{k}) = 0$ can produce all the information and the physical quantities regarding wave propagation in either a stable or an unstable system.

The main purpose of the present report is to provide a detailed study on the problem of wave radiation/propagation in an unstable system. Specifically, we examine the classification of instability, its mathematical criteria, and the computation of growth rates. All these entities are, in fact, contained in the dispersion equation. One method of recognizing them explicitly is to use the dispersion equation, causality or an appropriate initial condition, and the boundary conditions, and to perform a detailed analysis on both the complex ω plane and the complex $\vec{k}$ plane.

III   CLASSIFICATION OF INSTABILITIES

Basically, two distinct types of instability can occur when a wave interacts with an unstable system. The two types are termed absolute instability and convective instability. The distinction between them becomes apparent only after a spectrum of waves is superimposed by inverting the Fourier and Laplace transformations (details of which are given in the next section).

We consider an initially quiescent and uniform, time-invariant medium, which permits us to look for solutions such as $\exp j(\omega t - \vec{k} \cdot \vec{x})$ of the linearized wave equation. The frequency and wavenumber are connected through the dispersion equation $D(\omega, \vec{k}) = 0$. If $\vec{k}$ is real and $\mathrm{Im}\ \omega < 0$, this solution implies that the amplitude of the perturbation will increase exponentially with time for any $\vec{x}$. Hence, within the limit of the linearized theory, the system is unstable and no steady-state conditions appear possible. To examine the nature of increasing amplitude of the perturbation we note that the response $\vec{r}(t,\vec{x})$ can be expressed by

$$\vec{r}(t,\vec{x}) = (2\pi)^{-4} \int_L d\omega \int d\vec{k}\ \vec{A}(\omega,\vec{k}) e^{j(\omega t - \vec{k} \cdot \vec{x})} \tag{1}$$

where L denotes the Laplace path, and $\vec{A}(\omega,\vec{k})$, inversely proportional to $D(\omega,\vec{k})$, is the associated amplitude of the wave. As $t \to \infty$, Eq. (1) may take on one of the two forms illustrated in Figure 1. The disturbance may grow in time at every point in space; or, if the speed of the disturbance is sufficiently large, it may propagate away while growing with time and leave the spatial region behind unperturbed. The former situation is referred to as an absolute instability and the latter as a convective instability.

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The importance of making the above distinction should now be clear. If a system is convectively unstable, a steady state can exist, and hence a real frequency ω and the corresponding complex wavenumber-vector $\vec{k}(\omega)$ solutions of the dispersion equation. The original oscillatory wave is being amplified as it propagates along in the system. This effect will be totally washed out if an absolute instability is present. In the situation of absolute instability, the response grows monotonically with time, at any point in space, until it is ultimately limited by nonlinear effects.
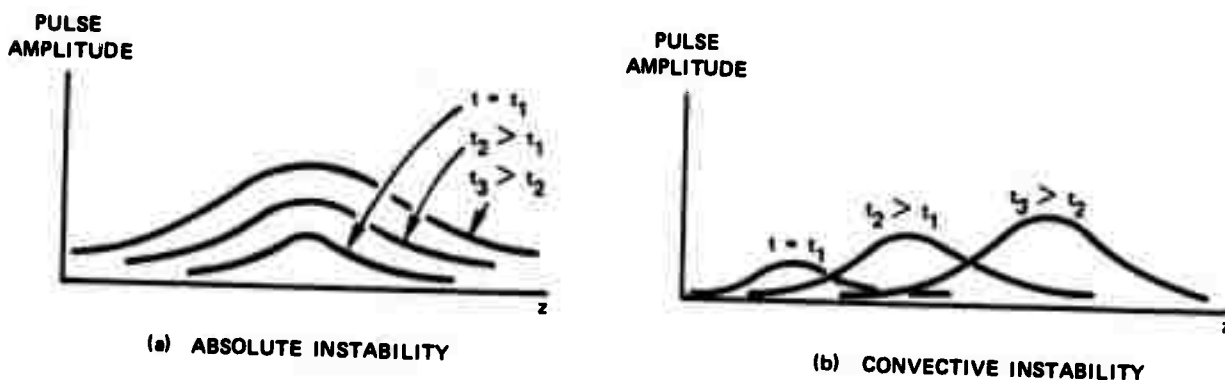

FIGURE 1   SKETCHES ILLUSTRATING ABSOLUTE AND CONVECTIVE INSTABILITIES

IV INSTABILITY ANALYSIS--CONFORMAL-MAPPING METHOD

One basic analysis that explains the instability theory and defines the various types of instabilities in an unstable system is accomplished by using the dispersion equation together with the appropriate initial and boundary conditions, and performing a conformal mapping from the complex ω plane onto the complex $\tilde{k}$ plane, or vice-versa. Our task in this section is to present a detailed study of the conformal mapping. During the process of the mapping, the growth rate for the wave can also be obtained when the system is unstable.

For simplicity, we shall consider only one-dimensional problems. The one-dimensional analysis actually covers an infinitely uniform system or a transverse eigenmode of an infinitely long transversely bounded system.

For elementary perturbations propagating as exp j $(\omega t - kz)$, the response of a linear system can be expressed by the following Laplace-Fourier inversing integral:

$$r(z,t) = \frac{1}{4\pi^2} \int_L d\omega \int_{-\infty}^{\infty} dk \frac{S(\omega,k)}{D(\omega,k)} e^{j(\omega t - kz)} \tag{2}$$

where

$S(\omega,k)$ = Laplace-Fourier transform of the source function $S(t,z)$

$D(\omega,k)$ = Dispersion relation of the system

L = Laplace contour in the complex ω plane.

The integration with respect to k in Eq. (2) is carried out along the real k line as shown in Figure 2. At this stage we assume that $S(\omega,k)$ is an entire function and also that $D(\omega,k) = 0$ has no branch points in

the complex k plane. Thus, the integrand of Eq. (2) has only pole singularities in the complex k plane. Using Cauchy's residue theorem, we perform a contour integration in the complex k plane by closing the k contour in the lower (or upper) half-plane for the response $z > 0$ (or $z < 0$). The appropriate k contours are illustrated in Figure 2 and the response is given by

$$r_>(z,\omega) = -j \sum_{n=1,2,\ldots} \frac{S(\omega,k_n)e^{j(\omega t - k_n z)}}{\left(\frac{\partial D}{\partial k}\right)_{k_n}} \quad , \quad z > 0 \qquad (3a)$$

$$r_<(z,\omega) = j \sum_{m=1,2,\ldots} \frac{S(\omega,k_m)e^{j(\omega t - k_m z)}}{\left(\frac{\partial D}{\partial k}\right)_{k_m}} \quad , \quad z < 0 \qquad (3b)$$

where k_n and k_m represent the distinct simple zeros of $D(\omega,k) = 0$ in the lower half-plane and the upper half-plane, respectively.

The above choice of k contours for $z > 0$ and $z < 0$ is subject to the radiation conditions at $|z| \to \infty$ (boundary conditions at infinity), and for $t > 0$, Eq. (3) represents outgoing waves propagating away from the source.

To determine the Laplace contour for integration with respect to ω, we impose the initial condition that response is zero for $t < 0$. Thus the Laplace contour should be chosen along a line Im $\omega = \omega_i = -\sigma (\sigma > 0)$ below the lowest pole of $r(z,\omega)$. The contour should then be closed in the upper half-plane for $t > 0$ and in the lower half-plane for $t < 0$, as illustrated in Figure 3. Since $r(z,\omega)$ has no poles in the region below the Laplace contour, there is no response for $t < 0$, as demanded by causality. For $t > 0$, it is clear from Eq. (2) and Figure 3 that integration along the large semicircle yields no contribution. Therefore, the integral with respect to ω can be evaluated by summing the residues at the poles.
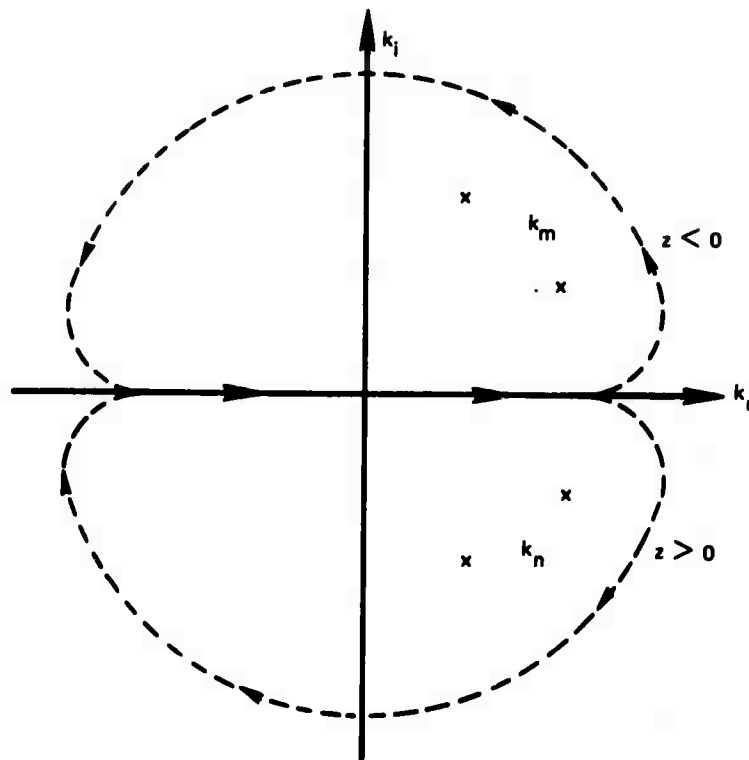
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**FIGURE 2   APPROPRIATE CONTOURS IN THE k-INTEGRATION,
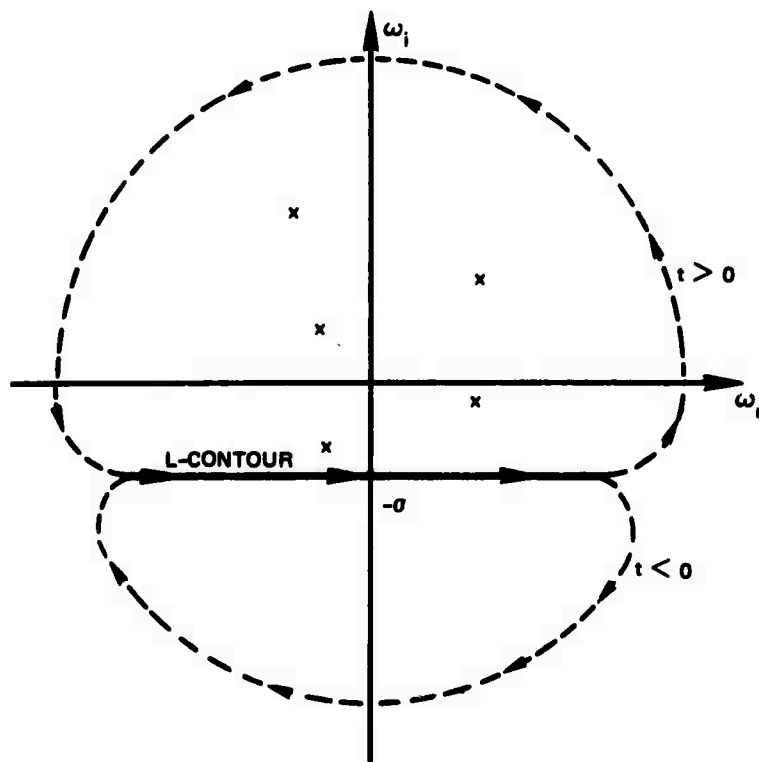WHERE x DESIGNATES THE POLES**

**FIGURE 3   APPROPRIATE CONTOURS IN THE LAPLACE INVERSION,
WHERE x DESIGNATES THE POLES**

It is now clear that when $r(z,\omega)$ has poles within the strip $0 \geq \omega_1 \geq -\sigma$, an instability in some sense may exist in the system. To investigate this instability, we use the dispersion equation $D(\omega,k) = 0$ and map the strip $0 \geq \omega_1 \geq -\sigma$ onto the complex k plane. From the types of loci for the corresponding poles in the complex k plane we can determine the nature of the instability.

Generally, one does not wish to study the response of the system for all time $(0 \leq t < \infty)$. It will suffice to know the behavior of $r(z,t)$ as $t \to \infty$. In the limit of $t \to \infty$, the response will ultimately be dominated by the fastest-growing component; hence, our interest should be focused on the pole closest to the line $\omega_1 = -\sigma$.

A.   Conformal Mapping of the ω Plane onto the k Plane

Consider the situation shown in Figure 4. Let $\omega_o = \omega_{or} + j\omega_{oi}$ $(\omega_{oi} < 0)$ be a pole below the real ω line. The corresponding poles in the k plane are determined from $D(\omega,k) = 0$, as shown in Figure 4(b). We now let ω_{oi} be less negative--i.e., move the Laplace contour toward the real ω line, as shown in Figure 4(a). In Figure 4(b), we indicate how the corresponding poles in the k plane move as ω_{oi} increases from $-\sigma$ toward zero. For the case of no double k root of $D(\omega,k) = 0$ considered here, two types of loci are possible: The loci may remain in their half-planes, or they may cross the real k line. We remark that as a pole crosses the real k line there will be a discontinuous jump in the values of $r_>(z,\omega)$ and $r_<(z,\omega)$. This is caused by the transfer of a residue to the other half-plane. The concept of analytic continuation should be used to derive the correct response-- i.e., if we wish to determine the response by integrating in a region above the Laplace contour $(\omega_1 = -\sigma$ line),(say, along the $\omega_1 = 0$ line), we should also deform the real k contour in such a way that it only includes residues at the poles that were in a given half-plane when values of ω on the Laplace contour were used. Thus, the analytic continuations of $r_>(z,\omega)$ and $r_<(z,\omega)$ are consequently obtained by following the deformed contour indicated in Figure 4(b).

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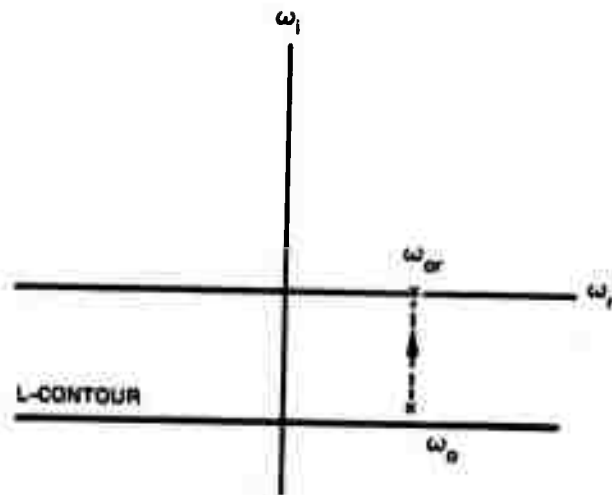
(a) THE PATH FROM ω_o TOWARD ω_{or}
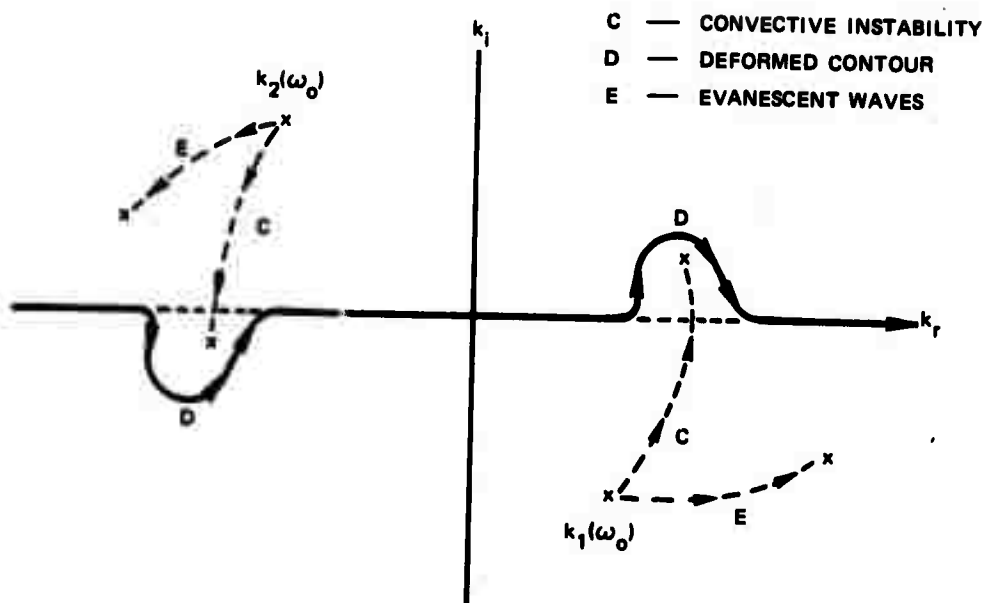
(b) LOCI OF POLES IN THE k PLANE AS ω_{oi} INCREASES TOWARD ZERO

FIGURE 4   LOCI OF POLES IN THE COMPLEX ω AND k PLANES

It is now clear how to define convective instability and amplifying waves. Several cases for a wave component with frequency $\omega = \omega_{or}$ giving rise to amplifying and/or evanescent waves are illustrated in Figure 4(b). To determine the maximum growth rate, the dispersion equation should be solved for all real ω to find the maximum value of Im k.

In the situation discussed above, the dispersion equation has no double roots of k. If in the process of moving the Laplace contour toward the real ω line, $D(\omega,k)$ has a double root of k at (ω_s, k_s). In the vicinity of this root we expand $D(\omega,k) = 0$:

$$D(\omega,k) \approx D(\omega_s,k_s) + \left(\frac{\partial D}{\partial \omega}\right)_s (\omega - \omega_s) + \left(\frac{\partial D}{\partial k}\right)_s (k - k_s) + \frac{1}{2}\left(\frac{\partial^2 D}{\partial k^2}\right)_s (k - k_s)^2$$

$$+ \left(\frac{\partial^2 D}{\partial \omega \partial k}\right)_s (\omega - \omega_s)(k - k_s) = 0 \qquad (4)$$

where the various derivatives are to be evaluated at (ω_s, k_s).

Since $D(\omega_s, k_s) = (\partial D/\partial k)_s = 0$ (the condition for a double root or a saddle point of k), Eq. (4) then yields two roots at

$$k_\pm = k_s \pm j\left[2(\omega - \omega_s)(\partial D/\partial \omega)_s/\left(\partial^2 D/\partial k^2\right)_s\right]^{1/2} \qquad . \qquad (5)$$

It is seen from Eq. (5) that in the limit of $\omega \to \omega_s$, the two poles $k_\pm$ merge with each other at k_s. This situation is illustrated in Figure 5.

Arbitrarily associating one root in the upper-half k plane and the other in the lower-half k plane, Eqs. (5) and (2) yield the identical response for $z > 0$ and $z < 0$:

$$r_>(z,\omega) = r_<(z,\omega) \approx \frac{S(\omega,k_s)e^{j\left(\omega t - k_s|z|\right)}}{\left[2\frac{\partial D}{\partial \omega}\frac{\partial^2 D}{\partial k^2}\right]_s^{\frac{1}{2}}(\omega - \omega_s)^{\frac{1}{2}}} \qquad . \qquad (6)$$

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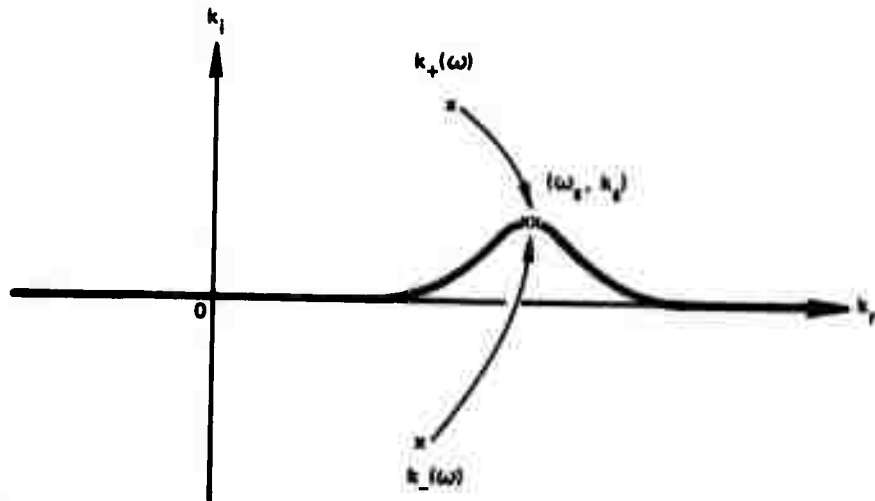
FIGURE 5   PINCHING OF THE CONTOURS OF $r_\leq(z,\omega)$ BY TWO POLES

It is seen from Eq. (4) or Eq. (6) that $\omega = \omega_s$ is a branch point in the ω plane and that the contribution at $\omega = \omega_s$ dominates the response. Note that as $\omega \to \omega_s$, Eqs. (5) and (6) not only give rise to the coalescence of two poles, but also result in the pinching of the contour in the k complex, as shown in Figure 5.

If this ω_s is the singularity with the fastest growth rate, we substitute Eq. (6) into Eq. (2) and perform the Laplace inversion. The asymptotic response of the system is given by

$$r(z,t) = \frac{S(\omega_s,k_s)}{\left[2\pi j \frac{\partial D}{\partial \omega}\frac{\partial^2 D}{\partial k^2}\right]_s^{\frac{1}{2}}} \frac{e^{j\left(\omega_s t - k_s |z|\right)}}{t^{\frac{1}{2}}} \quad , \quad \text{for all } z . \qquad (7)$$

Thus, the effect of the pinching due to the occurrence of a saddle point in the k complex plane makes the response infinite at $\omega = \omega_s$ (note that $\omega_{si} < 0$) for all z. This type of instability, involving merging

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poles in the k complex, is termed absolute instability. At any point in space, the response grows monotonically with time, until it is ultimately limited by nonlinear effects.

It may be remarked that coalescence of poles that do not pinch the contour gives a finite response (no absolute instabilities).

From Eq. (4), we note that $(\partial D/\partial \omega)_s \neq 0$ [since ω_s is not a double root of $D(\omega,k)$]. The condition $(\partial D/\partial k)_s = 0$ then results in the vanishing of group velocity $v_g = (\partial \omega/\partial k)_s$ at point (ω_s, k_s). Thus, in the situation of absolute instability, the energy of the response does not propagate but increases monotonically with time within the entire system.

B. <u>Conformal Mapping of the k Plane onto the ω Plane--Frequency Cusp</u>

An alternative procedure to determine the nature of instability is suggested by the fact that a saddle point at k_s in the k plane, with $\omega = \omega_s$ and $\omega_{si} < 0$, maps a branch point in the lower-half ω plane. Thus, one can now reverse the previous process of analytic continuation (in the k plane) by mapping the lines parallel to the Im k axis from the k plane onto the ω plane. Eq. (4) or (5), for the occurrence of absolute instability, can be inverted in the form:

$$\omega \approx \omega_s - \frac{1}{2}\left(k - k_s\right)^2 \left[\frac{\partial^2 D}{\partial k^2} \Big/ \frac{\partial D}{\partial \omega}\right]_s + \ldots \tag{8}$$

Equation (8) shows that one of these contours in the complex ω plane necessarily traces a cusp and thereby locates the desired frequency ω_s for the absolute instability.

However, we note that, from the previous saddle-point analysis, a saddle point that does not pinch the contours from the upper- and lower-half planes results in no absolute stability. In order to differentiate between the types of instability by this reverse mapping process, it is important to know the additional properties of a cusp to determine an absolute instability.

From the analysis in the previous section, we note that a saddle point k_s, which pinches the contours and results in an absolute instability, is produced by the merging of the two poles from the two opposite half-planes and the pinching of the contours. In fact, the two half k planes can be considered as two Riemann sheets in the k space when we trace the loci of the roots of $D(\omega,k) = 0$ in the k space, and the pinched contour can be regarded as a branch cut. The corresponding situations in the ω plane are as follows: (1) the saddle point at k_s maps into a branch point ω_s and in the vicinity of ω_s the contour forms a cusp, and (2) the cusp starts and terminates on two different Riemann sheets of the ω plane (corresponding to the pinching of the contours in the k plane).
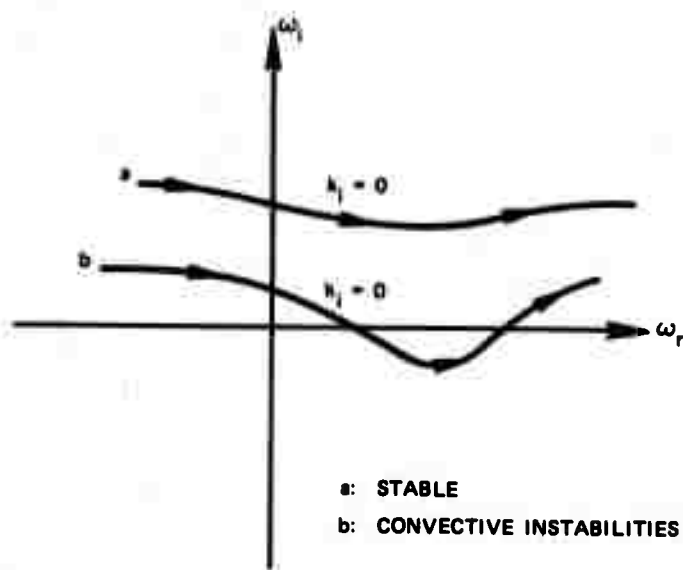
We now summarize the procedure as follows:

(1) We take the dispersion equation and map the real k line into the ω plane. In general, a discrete number of modes $\omega_n(k)$ is thus obtained.

(2) If $\mathrm{Im}\,\omega_n(k) \geqslant 0$ for n and k, the system is stable and supports only evanescent waves.

(3) If $\mathrm{Im}\,\omega_n(k) < 0$ for some n and range of k--say, $k_A \leqslant k \leqslant k_B$-- the system is unstable. To determine the instability type, we map lines, k_r = constant, with abscissas $k_A \leqslant k \leqslant k_B$ into the ω plane and find a cusp that:

  (a) Is centered at a branch point ω_s with $\mathrm{Im}\,\omega_s < 0$

  (b) Starts and terminates on two different Riemann sheets of the ω plane.

If the above conditions 3a and 3b are satisfied, the instability is absolute; otherwise it is convective.

A qualitative diagram for these contours in the ω plane is shown in Figure 6.

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a: STABLE

b: CONVECTIVE INSTABILITIES

(a) NO CUSPS

c: ABSOLUTE INSTABILITY (the path starts and terminates on two different sheets of the ω plane)

d: CONVECTIVE INSTABILITY (the entire path on the same sheet — ignorable branch point)

BRANCH CUT

ω_s (branch point)
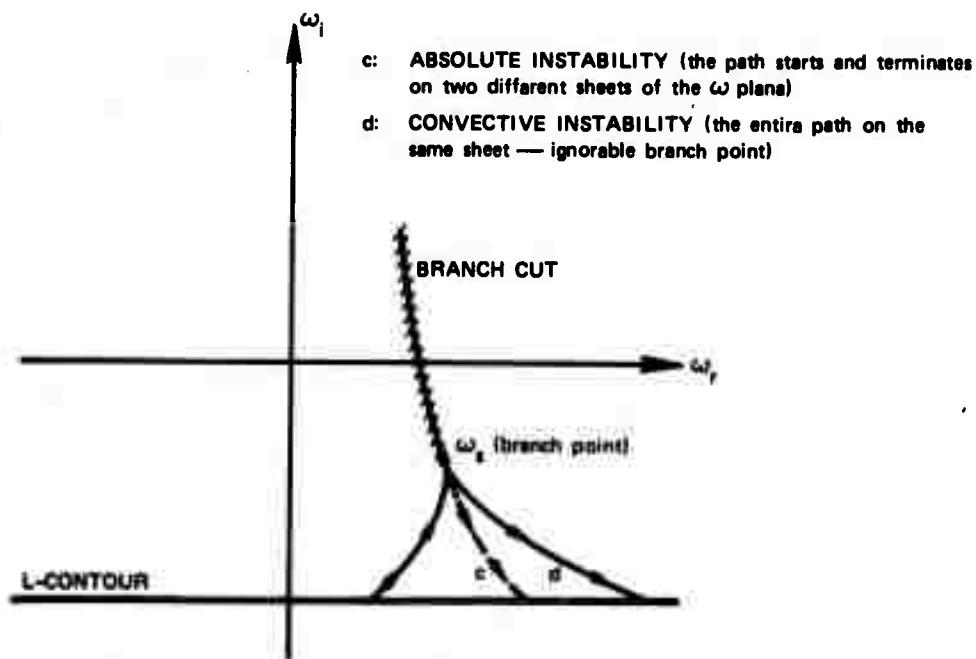
L-CONTOUR

(b) CUSP IN THE ω PLANE

FIGURE 6   QUALITATIVE ILLUSTRATIONS FOR REAL k LINE CONTOURS MAPPED IN THE COMPLEX ω PLANE

V   INSTABILITY ANALYSIS IN WARM-PLASMA SYSTEMS

In the previous analysis we assumed that $D(\omega, k) = 0$ has no branch points (and thus no branch cuts) in the complex k plane. It follows that there is only one form of the dispersion equation for the entire k complex. When kinetic theory is used to investigate the problems of wave interaction with warm plasma, the dispersion equation can have branch points in the complex k plane. The wave/warm-plasma system can then have more than one distinctly different dispersion equation, depending on the regions in the k plane. For example, the dispersion equations for the propagation of longitudinal waves in a collisionless, hot, isotropic plasma are given by

$$D_{\pm}(k,\omega) = 1 - \frac{\omega_p^2}{k^2} \fint_{-\infty}^{\infty} \frac{\partial f_o(v)/\partial v}{v - \omega/k} \, dv \pm j\pi \frac{\omega_p^2}{k^2} \left(\frac{\partial f_o(v)}{\partial v} \right)_{\omega/k} \tag{9}$$

where $f_o(v)$ denotes unperturbed velocity distribution in one direction, and $\fint$ denotes the principal part of the integral with the integration along the Im v axis (note that Im $v = v_i = (\omega/k)_i$).

The branch cut in the k plane associated with Eq. (9) is shown in Figure 7(a) with $D_+(k,\omega)$ valid above it and D_- valid below it. To calculate the response $r(z,\omega)$, one is faced with the problem of deciding how to carry out the k integration through the branch cut. In view of the branch cut not being extended to $k = 0$ (this is due to the fact that no particles have $v > c$, the speed of light in vacuum), one possible integration contour is illustrated in Figure 7(b). Now, as far as stability analysis is concerned, it is clear that the branch line lies

below the k real axis for k_r (=Re k) > 0. Thus, all of the considerations studied in Section IV-A, "Conformal Mapping ω onto the k Plane" for determining the instabilities are preserved if one uses $D_+(\omega,k)$ for $k_r > 0$ and $D_-(\omega,k)$ for $k_r < 0$.

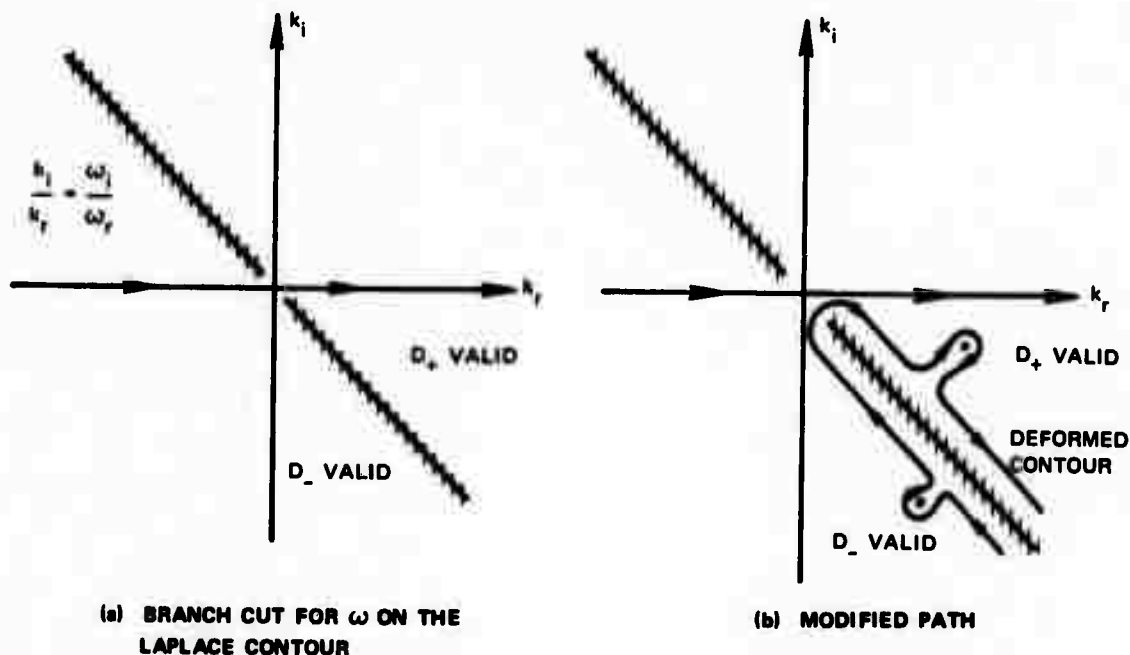(a) BRANCH CUT FOR ω ON THE LAPLACE CONTOUR

(b) MODIFIED PATH

FIGURE 7   BRANCH CUT AND MODIFIED PATH IN THE COMPLEX k PLANE

A useful criterion for determining a system being unstable has been developed by Penrose, and is known as the Penrose Criterion. To describe the Penrose Criterion, we consider the dispersion equation for one-dimensional longitudinal waves in a uniform non-Maxwellian plasma:

$$k^2 = Z\left(\frac{\omega}{k}\right) = \omega_p^2 \int_{-\infty}^{\infty} \frac{\partial f_o(v)/\partial v}{v - \omega/k} \, dv \quad . \tag{10}$$

The Penrose Criterion states that the system can be unstable only if $Z(\omega,k)$ can be real and positive. To prove the above statement, we assume that perturbations vary like exp $j\,(\omega t - kz)$. It was shown in Section IV that for a system to be unstable, the dispersion equation must have a solution with Im $\omega < 0$ for real k.

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Let k be real in Eq. (10). It follows that instability exists only if $Z(\omega/k)$ takes a real positive value somewhere in the lower half of the ω/k plane. Note that if we assume that the waves vary like exp $-j(\omega t - kz)$, then $Z(\omega/k)$ takes a real positive value somewhere in the upper half-plane, permitting the existence of instability.

Equation (10) defines a mapping from the v plane to the Z plane. If instability is to exist in the plasma for a given distribution $f_o(v)$, solutions of Eq. (10) must exist for which Z is real and non-negative when Im v < 0 (k is taken to be positive). It is then clear that the condition for instability is just that the region of the mapping of the lower half of the v plane upon the Z plane by means of Eq. (10) must contain some portion of the positive real axis.

The above criterion is particularly useful for the determination of the marginal instability (Im v → $_{\bullet}$0) for an unstable system.

VI PROCEDURE FOR ANALYZING INSTABILITIES

It is useful to summarize a few steps in the procedure for analyzing instabilities [assuming that the waves vary like exp $j(\omega t - kz)$]:

(1) Given a dispersion equation, the first step is to look for ω-complex roots for k real. If there are no roots with $\omega_i < 0$, the system is stable and supports only undamped or evanescent waves. Under these conditions, it is legitimate to solve for k-complex roots with ω real.

(2) If there are roots with $\omega_i < 0$ in (1), the system is unstable. We can perform one of the two mappings to determine the type of instability: (1) The dispersion equation can be used to map ω into the k plane. Plots such as Figure 8 may be obtained showing the saddle points and poles for discrimination between absolute and convective instabilities. (2) The dispersion equation can be used to map the real k axis into the ω complex. Plots such as shown in Figure 9 may be obtained to determine the absolute instability.

(3) Conformal mapping is usually a very tedious process. It is thus useful to apply some fairly crude checks to the dispersion equation. One method is to consider the solutions of the dispersion equation for Im $\omega \rightarrow -\infty$ and 0. If all the k roots of $D(k, \omega_r - j\infty) = 0$ lie on one of the half k planes and the k roots of $D(k, \omega_r - j0) = 0$ lie on the same or opposite half-plane, there is no possibility of the poles pinching the contours. Thus the system is either stable or convectively unstable. On the other hand, when $D(k, \omega_r - j\infty) = 0$ has k roots

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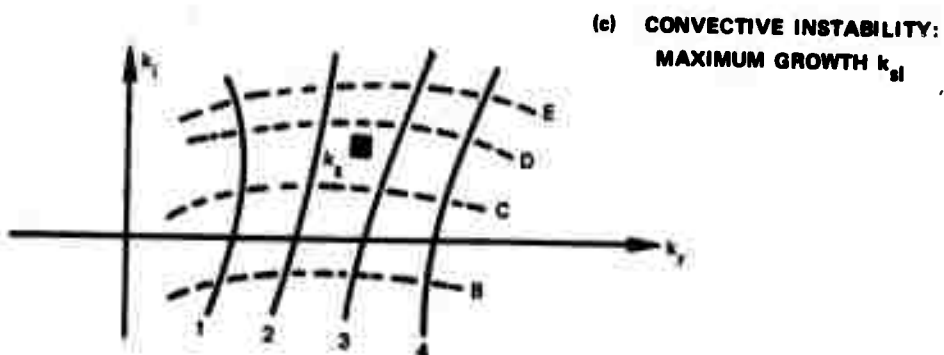(a) CONTOURS IN THE ω PLANE

(b) ABSOLUTE INSTABILITY
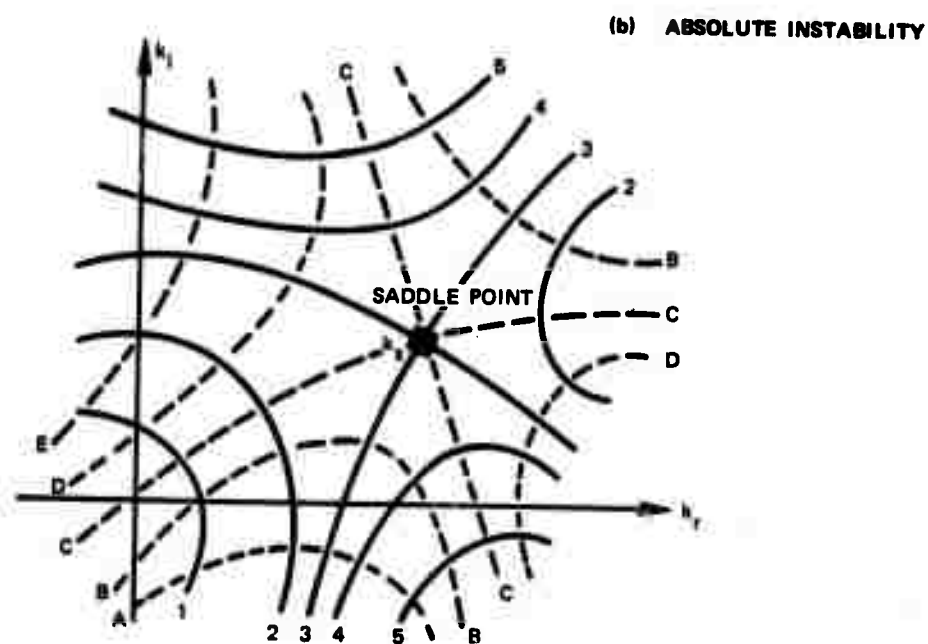
(c) CONVECTIVE INSTABILITY:
MAXIMUM GROWTH k_{si}
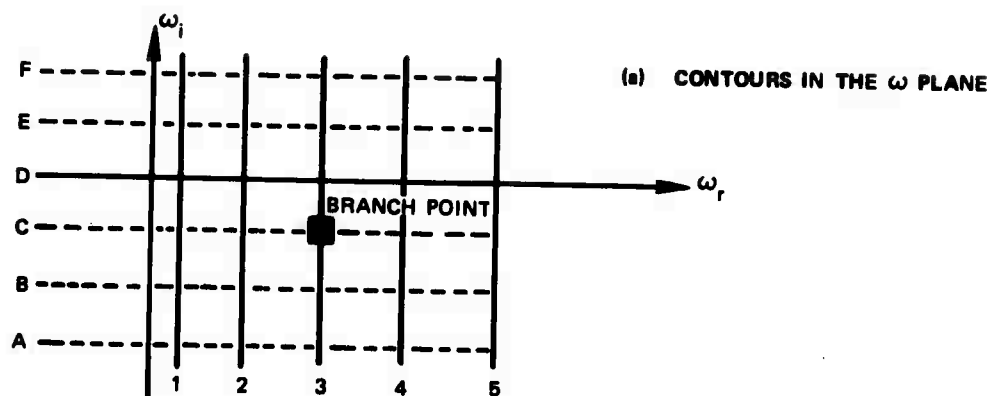
FIGURE 8   QUALITATIVE ILLUSTRATIONS FOR MAPPING THE LOWER-HALF
ω PLANE ONTO THE k PLANE TO LOCATE THE INSTABILITIES
AND DETERMINE THE GROWTH RATES

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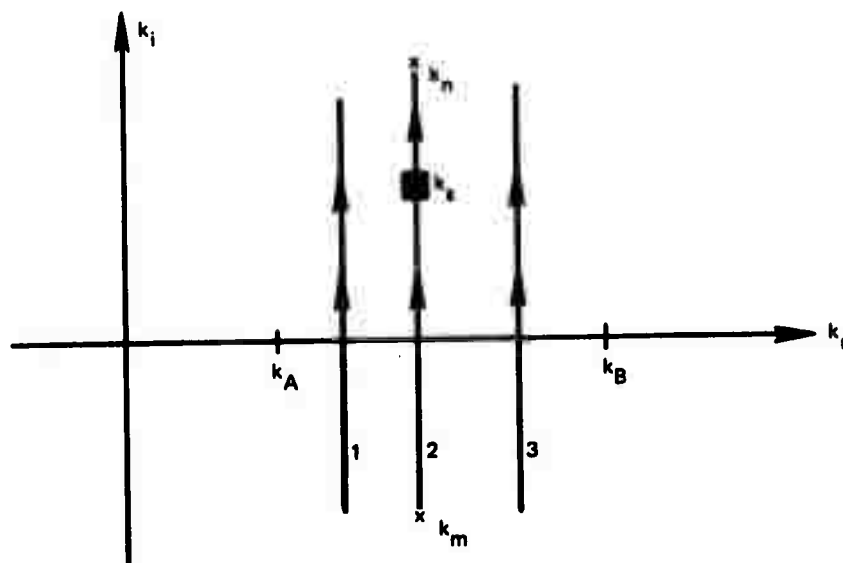
(a) k_r = CONSTANT LINES, WITH ABSCISSAS $k_A < k_r < k_B$

a = CONTOUR FOR k_i = 0, SHOWING
$\omega_i < 0$ FOR $k_A < k_r < k_B$
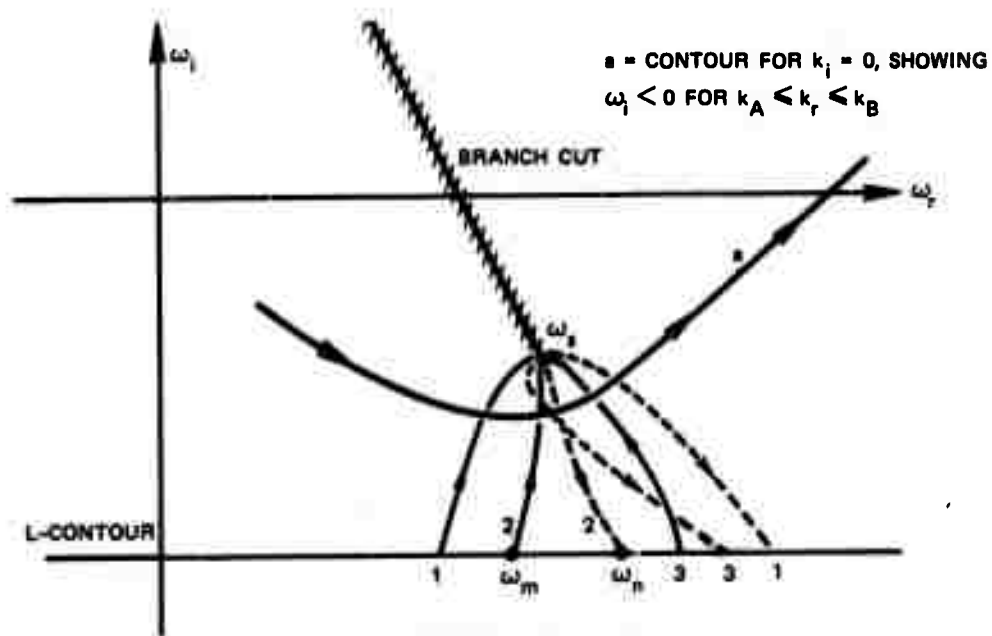
(b) ABSOLUTE INSTABILITY

FIGURE 9 QUALITATIVE ILLUSTRATIONS OF MAPPING OF THE k COORDINATES, $k_A < k_r < k_B$, TO DETERMINE ABSOLUTE INSTABILITY

on different half-planes, it is possible for the poles to
pinch the contours as Im ω increases from $-\infty$. Consequently,
we check for double roots of k of $D(k,\omega) = 0$. If a double
root k_s exists for some ω_s with Im $\omega = \omega_{si} < 0$, then there
is an absolute instability at (ω_s, k_s).

VII   EFFECTS OF DRIVING SOURCE

In the previous sections we have conducted the instability analysis under the assumption that the transformed source function $S(\omega,k)$ is an entire function. Thus the basic characteristics of the response are determined by the singularities of the dispersion equation alone. We now consider the response of an unstable system to a continuing source at spatial origin $(z = 0)$ by taking the source function $S(t,z) = \delta(z)$ $\exp j\omega_e t$. When ω_e is real, it can be regarded as a Fourier component of a periodic excitation. On the other hand, when $\omega_e = \omega_{er} + j\omega_{ei}$ is complex, it can be regarded as an external excitation of increasing $(\omega_{ei} < 0)$ or decreasing $(\omega_{ei} > 0)$ amplitude according to $(\exp - \omega_{ei} t)$ $\exp j\omega_{er} t$.

The analysis described in the previous sections for evaluating the asymptotic response proceeds as before. However, in evaluating the Laplace integral of Eq. (2), we must take account of the singularity of $S(\omega,k) = \left[j(\omega - \omega_e) \right]^{-1}$ (i.e., the pole at $\omega = \omega_e$). It is clear that if ω_e lies above the lowest relevant branch point (ω_s,k_s) of $D(\omega,k) = 0$ (i.e., if $\omega_{ei} > \min \omega_{si}$), then the asymptotic response will still be given by Eq. (7). On the other hand, if ω_e lies below the lowest relevant branch point (i.e., if $\omega_{ei} < \min \omega_{si}$), then the asymptotic response will be governed by the applied source, which can be evaluated by the use of Eqs. (2) and (3):

$$r(z,t) = r(z,\omega_e)\, e^{j\omega_e t}$$

$$t \to \infty \tag{11}$$

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where $r(z, \omega_e)$ is given either by (3a) or (3b) with exp $j\omega t$ suppressed, and the summation in Eq. (3) is now over the modes $k_n(\omega_e)$ or $k_m(\omega_e)$. Note that $r_>(z, \omega_e)$ or $r_<(z, \omega_e)$, depending on whether $z > 0$ or $z < 0$, should be analytically continued to determine whether a particular root $k(\omega_e)$ appears in the response and, hence, from the sign of $k_i(\omega_e)$, whether it grows or decays in space away from the source.

Hitherto, in this type of analysis, ω_e has been taken as real, corresponding to steady excitation, and the analysis is applicable only for systems that are not absolutely unstable. In this case the argument given above leads to criteria for deciding on which side of the driving point the various waves $k(\omega_e)$ appear, and hence, from the sign of k_i, whether they are amplifying or damped waves. The extension to complex ω_e is needed for bounded systems and is not considered in this report.

In the above context, it is physically clear that even if the system is absolutely unstable, provided the drive increases more rapidly than the fastest growing absolute instability, the asymptotic response will be governed by the source and not by the natural response of the system. On the other hand, if the system is not absolutely unstable, the asymptotic response will again be governed by the source, even though it is a decreasing one, provided it decays less rapidly than the slowest-decaying natural response of the system. In either case the criteria usually employed for real ω_e, for determining whether the roots $k(\omega_e)$ appear in the response for $z < 0$ or $z > 0$, may still be employed for complex ω_e.

VIII CONCLUDING REMARKS

In the present report we have presented an elementary but detailed analysis on the theory of wave instability. Our analysis is based on a conformal mapping through the use of the dispersion equation. This method is by no means the only one to analyze the instabilities. Other approaches such as those based on an energy principle or the semi-quantum mechanical approach can also be useful in examining the instabilities of a wave system. These two approaches are particularly useful to investigate the instabilities for a nonlinear system. Nevertheless, the conformal-mapping method presented here is commonly known to most scientists and engineers and is most easily appreciated.

The distinction between absolute and convective instabilities has been made clear by now. We should emphasize the importance of applying instability criteria, rather than blindly solving dispersion equations for an arbitrary choice of complex k, real ω, or complex ω, real k.

In addition to the two variables ω and k, a dispersion equation usually contains a number of parameters. These parameters, based on the models assumed, are the mathematical representation of certain physical quantities of the system--e.g., finite boundaries, weak inhomogeneity, and collisions. Under the assumption of a wave-system model, the first step is to derive the dispersion equation $D(\omega,k,\alpha_i) = 0$, where α_i represent the various parameters.

To investigate the instabilities of the system, we can perform the conformal mappings through the use of $D(\omega,k,\alpha_i) = 0$. Since the solutions of $D(\omega,k,\alpha_i) = 0$ and the various loci in the mapping process

depend on the parameters α_i, it may be possible to control or adjust certain physical quantities of the system in order to obtain the desired instability for the wave growth. This point is particularly important in the design of plasma-injection experiments for proving certain feasible applications.

ACKNOWLEDGMENT

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